

LINEAR DEPENDENCE AND RANK

Dependence and Independence

We recognize that when a redundant equation is present in a system of equations, the matrix of coefficients is singular, the determinant is zero. This is one definition of a *linearly dependent* set of equations.

When none of the equations in a system are redundant with any other equation within the system, the system of equations is *linearly independent*.

Consider a vector $\underline{z} = a\underline{x} + b\underline{y}$, where \underline{z} is a linear function of \underline{x} and \underline{y} , or linearly dependent on \underline{x} and \underline{y} .

When we have used vector addition or vector subtraction in the past, we were creating linear combinations of vectors where the scalars were all equal to +1 or -1.

If we employed $a = 2$ and $b = -1$, where $\underline{x} = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$ and $\underline{y} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, then

$$\underline{z} = a\underline{x} + b\underline{y} = 2 \begin{bmatrix} 4 \\ 6 \end{bmatrix} - 1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 12 \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 11 \end{bmatrix}$$

Formally, a set of vectors, $\underline{a}_1, \underline{a}_2, \dots, \underline{a}_n$ is linearly dependent if there exists a set of scalars (not all zero) where:

$$c_1\underline{a}_1 + c_2\underline{a}_2 + \dots + c_n\underline{a}_n = \underline{0}.$$

If no set of constants exist to satisfy this condition, the set of vectors is linearly independent. For the above condition to be true, at least one of the vectors must be redundant with another, it can be expressed as a linear combination of the others. Linear dependence implies redundancy.

Generally, for a set of vectors, $\underline{a}_1, \underline{a}_2, \dots, \underline{a}_n$ and n scalars, c_1 through c_n , a linear combination of vectors can be computed: $c_1\underline{a}_1 + c_2\underline{a}_2 + \dots + c_n\underline{a}_n$.

Singularity

Recall that in a singular matrix, the determinant of the matrix is zero. In this case, one or more of the rows or columns is linearly dependent; there exists a set of scalars that satisfies our definition.

Earlier we saw that $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$? $|\mathbf{A}| = (2)(1) - (2)(1) = 0$.

In addition, there exists a set of scalars, $c_1 = 2$ and $c_2 = -1$ that when multiplied by the vectors of rows in \mathbf{A} , will yield a null vector:

$$c_1 \underline{a}_1 + c_2 \underline{a}_2 = \underline{0} \quad 2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} - 1 \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} - \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

In this case, \mathbf{A} is singular ($|\mathbf{A}| = 0$) where the rows are linearly dependent. The columns are similarly dependent since they are equal.

There is a theorem in linear algebra that suggests that any n-element vector can be written as a linear combination of n linearly independent n-element vectors.

Finding such vectors can be difficult, but we rely on this theorem in statistics.

Group Exercises:

Which of the following matrices have linearly dependent rows?

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} 2 & 3 & 8 \\ 15 & 5 & 9 \\ 6 & 9 & 24 \end{bmatrix}$$

Rank

The rank of a matrix describes its linear independence. It describes the dimensionality of the space. It is defined for any square or rectangular matrix such that

$$\begin{aligned} \text{rank}(\mathbf{A}) &= \text{number of linearly independent rows of } \mathbf{A} \\ &= \text{number of linearly independent columns of } \mathbf{A}. \end{aligned}$$

The number of linearly independent rows of a matrix is always equal to the number of linearly independent columns.

For \mathbf{A} of $n \times p$, the maximum possible rank is the smaller of n and p , where \mathbf{A} is considered to be of *full rank*. When the rank is smaller than both n and p , the matrix is of *deficient rank*. Thus, only a square matrix can be of full rank – rectangular matrices will always have a rank less than either n or p .

$$0 < \text{rank} \leq \min(n, p).$$

When a matrix is the product of two other matrices, $\mathbf{C} = \mathbf{AB}$, its rank cannot be greater than the smaller rank of the two matrices:

$$\text{Rank } \mathbf{C} \leq \min (\text{rank } \mathbf{A}, \text{rank } \mathbf{B}).$$

Example:

$$\mathbf{A}_{3 \times 3} = \begin{bmatrix} 1 & 3 & 5 \\ 1 & 2 & 3 \\ 2 & 5 & 8 \end{bmatrix} \quad \text{Rank} = 2, \text{ deficient rank; row 3} = \text{row 1} + \text{row 2}$$

$$|\mathbf{A}| = 16 + 18 + 25 - 20 - 15 - 24 = 0$$

So, \mathbf{A} is singular, no unique solution; inverse does not exist.

Within \mathbf{A} , there is a 2×2 matrix that is nonsingular

$$\mathbf{B} = \begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix}, \text{ where } |\mathbf{B}| = (1)(2) - (1)(3) = -1$$

So, rank of $\mathbf{A} = 2$

If \mathbf{A} is nonsingular ($|\mathbf{A}| \neq 0$), then

1. \mathbf{A} has an inverse
2. $r(\mathbf{A}) = n$, or \mathbf{A} is full rank
3. the rows of \mathbf{A} are linearly independent
4. the columns of \mathbf{A} are linearly independent

Rank & the Solution of Equations

To evaluate the independence of a system of equations, we need to know the rank of the coefficient matrix and the augmented matrix.

Consider our original system of linear equations:

$$x_1 + x_2 + x_3 = 5$$

$$2x_1 - 1x_2 + 6x_3 = 12$$

$$x_1 + 3x_2 + 5x_3 = 17$$

$$\text{Coefficient matrix } \mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 6 \\ 1 & 3 & 5 \end{bmatrix} \text{ with augmented matrix } \mathbf{A|y} = \begin{bmatrix} 1 & 1 & 1 & 5 \\ 2 & -1 & 6 & 12 \\ 1 & 3 & 5 & 17 \end{bmatrix}$$

A theorem in linear algebra states that a system of equations is independent (consistent) if the rank of the coefficient matrix is the same as the rank of the augmented matrix,

Where $r(\mathbf{A}) = r(\mathbf{A|y})$.

In our case, $|\mathbf{A}| = -5+6+6+1-18-10 = -20$

- this tells us that A is nonsingular
- $r(\mathbf{A}) = 3$, A is full rank
- $r(\mathbf{A|y})$ must be at least as large as $r(\mathbf{A})$
- since $\mathbf{A|y}$ is 3×4 , the largest square submatrix is 3×3
- $r(\mathbf{A|y})$ cannot exceed 3
- $r(\mathbf{A|y}) = r(\mathbf{A}) = 3$; the system is consistent.

Since $|\mathbf{A}| \neq 0$, \mathbf{A}^{-1} exists.

Recall, $\mathbf{A}\underline{x} = \underline{y} \rightarrow \underline{x} = \mathbf{A}^{-1}\underline{y}$

$$\mathbf{A}^{-1} = \begin{bmatrix} 1.15 & .1 & -.35 \\ .2 & -.2 & .2 \\ -.35 & .1 & .15 \end{bmatrix} \text{ so, } \underline{x} = \mathbf{A}^{-1}\underline{y} = \begin{bmatrix} 1.15 & .1 & -.35 \\ .2 & -.2 & .2 \\ -.35 & .1 & .15 \end{bmatrix} \begin{bmatrix} 5 \\ 12 \\ 17 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

Another theorem of linear algebra provides a way to evaluate whether a consistent set of equations has an infinite number of solutions: a consistent system of n equations in n unknowns has a unique solution if the rank of the coefficient matrix is equal to its order, where $r(\mathbf{A}) = n$.

In the example here, there are three equations and three unknowns and the rank of the coefficient matrix, \mathbf{A} , is 3. The solution we have computed is unique.

Another way to state this is:

A consistent system of equations where \mathbf{A} is of order n has a unique solution if and only if \mathbf{A}^{-1} exists.

This is important in applications where we have the same number of unknowns as equations.

Generalization of Full Rank and Unique Solutions

When the number of equations equals the number of unknowns and $|\mathbf{A}| \neq 0$, there is a unique solution.

When \mathbf{A} is full rank, a unique solution exists for $\mathbf{A} \underline{x} = \underline{y}$.

This is true because:

1. if $(|\mathbf{A}| \neq 0)$, $r(\mathbf{A}) = n$ (from the definition of rank)
2. if $r(\mathbf{A}|\underline{y}) = r(\mathbf{A})$, the equations are consistent (from first theorem)
3. if the equations are consistent and $r(\mathbf{A}) = n$, there is a unique solution.

Rank & Regression

Recall our solution to the general linear equation:

$$\begin{aligned} \underline{y} &= \mathbf{X} \underline{b} + \underline{e} \\ \mathbf{X}' \underline{y} &= \mathbf{X}' (\mathbf{X} \underline{b} + \underline{e}) \\ &= \mathbf{X}' \mathbf{X} \underline{b} + \mathbf{X}' \underline{e} = \mathbf{X}' \mathbf{X} \underline{b} \\ \underline{b} &= (\mathbf{X}' \mathbf{X})^{-1} \mathbf{X}' \underline{y} \end{aligned}$$

If the dimensions of \mathbf{X} are $m \times n$ and the rank of $\mathbf{X}' \mathbf{X}$ is less than n , the inverse of $\mathbf{X}' \mathbf{X}$ does not exist and there is no unique solution.

The rank of a product matrix cannot be larger than the smaller of the ranks of either matrix.

$$\text{Rank } \mathbf{AB} \leq \min(\text{rank } \mathbf{A}, \text{rank } \mathbf{B}).$$

Recall our example

$$\begin{aligned} Y_1 &= b_{11}X_1 + b_{12}X_2 + b_{13}X_3 + E_1 \\ Y_2 &= b_{21}X_1 + b_{22}X_2 + b_{23}X_3 + E_2 \\ Y_3 &= b_{31}X_1 + b_{32}X_2 + b_{33}X_3 + E_3 \\ &\vdots \\ Y_{100} &= b_{100,1}X_1 + b_{100,2}X_2 + b_{100,3}X_3 + E_{100} \end{aligned}$$

$$\text{Where } \underline{y} = \mathbf{X} \underline{b} + \underline{e}$$

with the dimensions: \underline{y} (100×1); \mathbf{X} (100×3); \underline{b} (3×1); \underline{e} (100×1);

and solution $\underline{b} = (\mathbf{X}' \mathbf{X})^{-1} \mathbf{X}' \underline{y}$,

What must the rank of \mathbf{X} be so we can solve for \underline{b} ?

The rank of \mathbf{X} must be 3.

The rank of the product matrix cannot be greater than the smaller of the ranks of the original matrices.

The product matrix $\mathbf{X}'\mathbf{X}$ must have an inverse so it must be full rank (3).

The three columns of \mathbf{X} must be linearly independent.