DESIGN MODELS

Design models accommodate qualitative, categorical variables.

Consider a model for three means, $\bar{y}_1$, $\bar{y}_2$, and $\bar{y}_3$.

$$\bar{y}_1 = \mu + \alpha_1 + e_1$$
$$\bar{y}_2 = \mu + \alpha_2 + e_2$$
$$\bar{y}_3 = \mu + \alpha_3 + e_3 \qquad \rightarrow \qquad \bar{y}_i = \mu + \alpha_i + e_i$$

$$\bar{y}_i = \mu + \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + e_i$$

$$\begin{aligned} \bar{y}_1 &= 1 \quad 1 \quad 0 \quad 0 \\ \bar{y}_2 &= 1 \quad 0 \quad 1 \quad 0 \\ \bar{y}_3 &= 1 \quad 0 \quad 0 \quad 1 \end{aligned}$$

$$\begin{bmatrix} \bar{y}_1 \\ \bar{y}_2 \\ \bar{y}_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mu \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} + \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}$$

$\underline{\bar{y}} = \mathbf{X}\,\underline{\alpha} + \underline{e}$ where $\mathbf{X}$ is the design matrix

In this case, $\underline{\hat{\alpha}} \neq (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\underline{y}$ since $|\mathbf{X}'\mathbf{X}| = 0$

In all design models, $|\mathbf{X}'\mathbf{X}| = 0$; design models are of deficient rank.

In our case, column 1 = column 2 + column 3 + column 4.

Example/

		α: Type of Twins	
		Monozygotic	Dizygotic
β: Gender	Male	$\bar{y}_{11}$	$\bar{y}_{12}$
	Female	$\bar{y}_{21}$	$\bar{y}_{22}$

We have an additive main class model; containing only main effects.

$$\begin{bmatrix} \bar{y}_{11} \\ \bar{y}_{12} \\ \bar{y}_{21} \\ \bar{y}_{22} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mu \\ \alpha_1 \\ \alpha_2 \\ \beta_1 \\ \beta_2 \end{bmatrix} + \underline{e}$$

$$\underline{\bar{y}} = \mathbf{A}\, \underline{\xi} + \underline{e}$$

Since $\mathbf{A}$ is of deficient rank, we cannot invert the matrix to solve the equations.

We can "reparameterize" the model to solve the deficiency problem.

Add equations: $$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} = D \rightarrow \begin{matrix} 0 = \alpha_1 + \alpha_2 \\ 0 = \beta_1 + \beta_2 \end{matrix}$$

Which suggests that $\Sigma\alpha_i = 0$ & $\Sigma\beta_i = 0$; ANOVA assumptions

This results in:

$$\begin{bmatrix} \underline{y.} \\ \cdots \\ \underline{0} \end{bmatrix} = \begin{bmatrix} A \\ \cdots \\ D \end{bmatrix} \xi + \begin{bmatrix} \underline{e} \\ \cdots \\ \underline{0} \end{bmatrix}$$

$$\underline{\hat{\xi}} = \left(\begin{bmatrix} A \\ \cdots \\ D \end{bmatrix}' \begin{bmatrix} A \\ \cdots \\ D \end{bmatrix} \right)^{-1} \begin{bmatrix} A \\ \cdots \\ D \end{bmatrix}' \begin{bmatrix} \underline{y.} \\ \cdots \\ \underline{0} \end{bmatrix}$$

This may not be the "best" solution because the introduction of $\underline{0}$ is arbitrary, although it is the "classical" ANOVA method.

The ideal method of reparameterization involves careful selection of the augmenting matrix. This allows us to test other parameters simultaneously as a component of the design.

$\underline{\bar{y}} = \mathbf{A}\,\underset{\sim}{\xi} + \underline{e}$ → with dimensions: (n×1) = (n×m)(m×1)+(n×1), where n = # of cells in design

Let $\mathbf{A}$ (n×m) be of rank l, where $l \leq$ m.

Let $\mathbf{L}$ be of rank l and linearly dependent on rows of $\mathbf{A}$.

$$\text{Rank}\begin{bmatrix} A \\ \cdots \\ L \end{bmatrix} = \text{Rank}(\mathbf{A}) = \text{Rank}(\mathbf{L}) = l$$

The key to the whole process is choosing L.

The best reparameterization for solving equations involving a design matrix includes a set of a priori contrasts.

Consider a one-way design with 4-levels:

$$\underline{\bar{y}} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \mu \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix} + \underline{e}$$

Examples of many possible Ls includes the following:

Simple Contrasts: $$L = \begin{bmatrix} 1 & 1/4 & 1/4 & 1/4 & 1/4 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} \mu + \dfrac{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4}{4} \\ \alpha_1 - \alpha_4 \\ \alpha_2 - \alpha_4 \\ \alpha_3 - \alpha_4 \end{bmatrix}$$

Helmert Contrasts $\quad L = \begin{bmatrix} 1 & 1/4 & 1/4 & 1/4 & 1/4 \\ 0 & 1 & -1/3 & -1/3 & -1/3 \\ 0 & 0 & 1 & -1/2 & -1/2 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} \cdots \\ \alpha_1 - \frac{\alpha_2 + \alpha_3 + \alpha_4}{3} \\ \alpha_2 - \frac{\alpha_3 + \alpha_4}{2} \\ \alpha_3 - \alpha_4 \end{bmatrix}$

Helmert contrasts are also orthogonal contrasts.

You can also create contrasts to assess your own specific research comparisons or a priori comparisons.