Vectors

Real numbers are called *scalars*. *Scalar arithmetic* is what we do with the operations of addition, subtraction, multiplication, and division of real numbers.

Ex/ 2 + 3 = 5, 15 ÷3 = 5

A *vector* is an ordered (in some specific order) array (two or more) of numbers.

Each number in a vector is called an *element* or *component*.

Vectors are described in terms of the number of elements they have.

Ex/ (5, 4) is a two-element vector

A vector may be written as a row or a column; these become row vectors or column vectors.

Ex/ $\underline{a} = \begin{bmatrix} 5 \\ 4 \\ 7 \\ 2 \end{bmatrix}$ is a four-element column vector.

Vectors are noted by lower-case, underlined letters. Some texts note vectors as lower-case bold.

A row vector is noted by lower-case underlined letters with a prime and enclosed in parentheses. The vector $\underline{a}'$ is the *transpose* of $\underline{a}$; it has been changed from a column- to a row-vector.

Ex/ $\underline{a}' = (5, 4, 7, 2)$ is a four-element row vector.

The general form for a column vector with n-elements is:

$$\underline{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_i \\ \vdots \\ a_n \end{bmatrix}$$

This may describe a set of test score data on *n* students.

Group Exercises

Identify each as a row vector, column vector, or scalar.

$(a_1 + a_2 + a_3 + a_4)$

$$\begin{bmatrix} 520 \\ 640 \\ 780 \end{bmatrix}$$

13

(115, 129, 92, 89)

5×11

(0, 0, 0)

More on Vectors

Vectors may be equal to other vectors if they satisfy three conditions:

1. same number of elements
2. corresponding elements from each vector must be equal
3. vectors must be both row-vectors or both be column-vectors

Exercises

$$\begin{bmatrix} 5 \\ 6 \end{bmatrix} \begin{bmatrix} 6 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} 9 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 9 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 55 \\ 69 \\ 48 \end{bmatrix} (55, 69, 48)$$

Vector Operations

Addition and Subtraction

Vectors can be added together if they

1. have the same number of elements
2. are both row vectors or both column vectors

Vectors a and b are added together here to form vector c

Ex/ $\underline{a} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$ $\underline{b} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ $\underline{a} + \underline{b} = \begin{bmatrix} 5 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 5+2 \\ 2+3 \end{bmatrix} = \begin{bmatrix} 7 \\ 5 \end{bmatrix}$

In the general case, you can add n-element vectors as long as they meet the requirements above.

$$\underline{c} = \underline{a} + \underline{b} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_i \\ \vdots \\ a_n \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_i \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \\ \vdots \\ a_i + b_i \\ \vdots \\ a_n + b_n \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_i \\ \vdots \\ c_n \end{bmatrix}$$

Because in addition of vectors, each corresponding component is added, it satisfies the commutative property and associative property of real numbers:

$\underline{a} + \underline{b} = \underline{b} + \underline{a}$ and $\underline{a} + (\underline{b} + \underline{c}) = (\underline{a} + \underline{b}) + \underline{c} = \underline{a} + \underline{b} + \underline{c}$

$$\underline{a} + \underline{b} + \underline{c} = \begin{bmatrix} 5+2+1 \\ 2+3+0 \end{bmatrix} = \begin{bmatrix} 8 \\ 5 \end{bmatrix}$$

Multiplication of Vectors by a Scalar

When a vector is multiplied by a scalar, each component of the vector is multiplied by the value of the scalar. Consider multiplication of a 4-component vector by scalar λ = 3,

$$\underline{a} = \begin{bmatrix} 2 \\ 3 \\ 0 \\ -1 \end{bmatrix} \qquad \lambda = 3 \qquad \lambda\underline{a} = 3\begin{bmatrix} 2 \\ 3 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} (3)2 \\ (3)3 \\ (3)0 \\ (3)(-1) \end{bmatrix} = \begin{bmatrix} 6 \\ 9 \\ 0 \\ -3 \end{bmatrix}$$

This is true for any n-element vector being multiplied by a scalar, λ.

Scalars may be fractions, negative numbers, or unknowns.

$$\lambda\underline{a} = \begin{bmatrix} 2\lambda \\ 3\lambda \\ 0 \\ -\lambda \end{bmatrix}$$

Division by a scalar

Technically, division in matrix algebra is undefined. To divide a vector by a scalar, multiply by its reciprocal.

Students' scores on a midterm and final ranged from 32 to 98 out of a possible 100 points on each exam. The scores for 6 students are shown below for the midterm (X) and final (Y) exam.

$$\underline{x} = \begin{bmatrix} 56 \\ 64 \\ 32 \\ 88 \\ 90 \\ 79 \end{bmatrix} \qquad \underline{y} = \begin{bmatrix} 50 \\ 69 \\ 51 \\ 98 \\ 87 \\ 70 \end{bmatrix}$$

1. What is the total score for each student? Find $\underline{z} = \underline{x} + \underline{y}$.

2. What is the mean score for each student? Find $(1/2)\,\underline{z}$.

3. If the students took three exams, how would you write the computation to obtain the mean in vector notation?

4. If the mean score on the final was 60, find each student's deviation score employing vector notation.

Multiplication of Vectors

There are two types of vector multiplication, the inner product and the outer product. The inner product yields a scalar value while the outer product yields a rectangular array of values.

The inner product – scalar product

multiply corresponding elements of each vector and add the products together.

$$\underline{a} = \begin{bmatrix} 5 \\ 4 \\ 7 \\ 2 \end{bmatrix} \qquad \underline{b} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 2 \end{bmatrix}$$

The inner product is always written as the product of a row vector and column vector:

$$\underline{a}'\underline{b} = (5, 4, 7, 2) \begin{bmatrix} 1 \\ 0 \\ -1 \\ 2 \end{bmatrix} = (5)(1) + (4)(0) + (7)(-1) + (2)(2) == 5 + 0 - 7 + 4 = 2$$

This order must always be followed because $\underline{a}'\underline{b}$ does not equal $\underline{a}\ \underline{b}'$.
Also, it is not possible to multiply more than two vectors.
In general notation, the scalar product is

$$\underline{a}'\underline{b} = (a_1, a_2, \ldots, a_i, \ldots, a_n) \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_i \\ \vdots \\ b_n \end{bmatrix} = (a_1 b_1, a_2 b_2, \ldots, a_i b_i, \ldots, a_n b_n) = \sum_{i=1}^{n} a_i b_i$$

Group Exercises

$$\underline{a}'\underline{b} = (1, 2, 4) \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$$

$\underline{a} = \begin{bmatrix} 5 \\ 2 \\ 3 \end{bmatrix}$ compute $\underline{a}'\underline{a}$

$\underline{a} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$, $\underline{b} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$ compute $\underline{a}'\underline{b}$ and $\underline{b}'\underline{a}$.

Special Vectors

Ones Vectors

A vector, $\underline{s}$, is called a "ones vector" when all elements are ones "1", $\underline{s} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

What rule emerges when we compute the scalar product of any vector and a vector of ones?

$$\underline{a} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, \underline{s} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \text{compute } \underline{a}'\underline{s} =$$

These are also called "sum vectors" and noted by $\underline{1}$ where $\underline{a}'\underline{s} = \underline{a}'\underline{1} = \sum_{i=1}^{n} a_i$

Elementary Vectors

The vectors $\underline{e}_1$, $\underline{e}_2$, $\underline{e}_3$ have special properties. Formulate a rule from the following exercise.

$$\underline{a} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, \underline{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \underline{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \underline{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \text{vectors with one value of 1 and all other zeros}$$

$\underline{a}'\underline{e}_1 =$
$\underline{a}'\underline{e}_2 =$
$\underline{a}'\underline{e}_3 =$

The rule that emerges is $\underline{a}'\underline{e}_i = a_i$, yielding a scalar product corresponding to the *i*th value specified by the elementary vector.

Null Vectors

Null vectors are vectors with all elements being zero. Find the general rule:

$$\underline{a} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, \underline{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \text{ where } \underline{a}'\underline{0} =$$